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On Δ_2 condition for density-type topologies generated by functions

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ABSTRACT

We discuss algebraic properties of density-type topologies generated by functions. We examine Δ_2 condition, similar to the condition considered in the theory of Orlicz spaces. We show that a topology $\mathcal{T}_f$ coarser than the density topology is invariant under multiplication by nonzero numbers if and only if f fulfills Δ_2 condition. We also show that for other topologies generated by functions, connections between algebraic properties of $\mathcal{T}_f$ and Δ_2 condition are more complicated.

1. Preliminaries

It is well known [10,14] that the notion of a density point of a measurable set $E \subset \mathbb{R}$ leads us to the notion of the density topology $\mathcal{T}_d$. The density topology is finer than the Euclidean topology. Also $\mathcal{T}_d$ is completely regular but not normal, and invariant under translations and multiplications by nonzero numbers. Modifying a little the definition of a density point we obtain a more general notion of f-density point and the related density-type topology generated by the function f (f-density topology).

Among f-density topologies there are: the density topology, topologies generated by sequences (cf. [6,5,3]), ψ-density topologies (cf. [12,13]) and others. Some of them have properties similar to properties of the density topology and some are quite different. In [2] it is proved that the fulfillment of separation axioms by a topology generated by a function f depends on the value of $\liminf_{x\to 0+} \frac{f(x)}{x}$.

In this paper we investigate mainly algebraic properties of topologies generated by functions. All such topologies are invariant under translations and symmetries. Invariantness under multiplication by nonzero numbers is connected with a condition similar to (Δ_2) condition considered in the theory of Orlicz spaces. Moreover, this condition allows us to simplify comparising topologies generated by functions.

Throughout the paper we shall denote by $\mathcal{L}$ the family of Lebesgue measurable subsets of the real line $\mathbb{R}$, and by $|E|$ the Lebesgue measure of E. Let $\mathcal{A}$ denote the family of all nondecreasing functions $f : (0,\infty) \to (0,\infty)$ such that $\lim_{x\to 0+} f(x) = 0$ and $\liminf_{x\to 0+} \frac{f(x)}{x} < \infty$. Fix $f \in \mathcal{A}$, $E \in \mathcal{L}$ and $x \in \mathbb{R}$. We say that x is a *right-hand f-density point* of E if

$$\lim_{h\to 0+} \frac{|(x, x+h) \setminus E|}{f(h)} = 0.$$

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By $\Phi_f^+(E)$ we denote the set of all right-hand f-density points of E. In the same way one may define *left-hand f-density points* of E and the set $\Phi_f^-(E)$. We say that x is an *f-density point* of E if it is a right and a left-hand f-density point of E. By $\Phi_f(E)$ we denote the set of all f-density points of E. If $x \in \Phi_f(\mathbb{R} \setminus E)$, we say that x is an *f-dispersion* point of E.

It is not difficult to prove that for any $E \in \mathcal{L}$ the sets $\Phi_f^+(E)$, $\Phi_f^-(E)$ and $\Phi_f(E)$ are measurable and the family $\mathcal{T}_f := \{E \in \mathcal{L}\colon\ E \subset \Phi_f(E)\}$ is a topology finer than the Euclidean topology on the real line (cf. [2]). Then $\mathcal{T}_f$ is called the *topology generated by a function f* or *f-density topology*. Observe that the condition $\liminf_{x\to 0+} \frac{f(x)}{x} < \infty$ is essential, because otherwise $\lim_{h\to 0+} \frac{|(x,x+h)\setminus E|}{f(h)} = 0$ for any $x \in \mathbb{R}$ and $E \in \mathcal{L}$. Clearly, if $f(x) = x$ then we obtain the "ordinary" density points and the density topology $\mathcal{T}_d$.

Different functions can generate the same operator and the same topology. But different operators generate different topologies. Namely, we have

Proposition 1. *([3, Proposition 4]) For each $f_1, f_2 \in \mathcal{A}$ the following conditions are equivalent*

(1) $\forall_{E\in\mathcal{L}} (0 \in \Phi_{f_1}^+(E) \Rightarrow 0 \in \Phi_{f_2}^+(E))$,
(2) $\forall_{E\in\mathcal{L}} (0 \in \Phi_{f_1}(E) \Rightarrow 0 \in \Phi_{f_2}(E))$,
(3) $\forall_{E\in\mathcal{L}} \Phi_{f_1}(E) \subset \Phi_{f_2}(E)$,
(4) $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$.

Now, we will formulate some natural connections between behaviour of functions f_1, f_2 and the inclusion $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$. A necessary and sufficient condition for $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$ will be formulated at the end of this section. Straightforwardly from the definition we obtain

Proposition 2. *([1]) If $f_1, f_2 \in \mathcal{A}$ and $\limsup_{x\to 0+} \frac{f_1(x)}{f_2(x)} < \infty$, then $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$.*

However, there exist functions generating the same topology, while $\limsup_{x\to 0^+} \frac{f_1(x)}{f_2(x)} = \infty$. An easy example of such a pair is the following: $\widetilde{f}_1(x) := \frac{1}{n!}$ for $x \in [\frac{1}{(n+1)!}, \frac{1}{n!})$, $n \in \mathbb{N}$, and $\widetilde{f}_2(x) := \frac{1}{n!}$ for $x \in (\frac{1}{(n+1)!}, \frac{1}{n!}]$, $n \in \mathbb{N}$ (see [3, Example 1]). Fortunately, if one of considered topologies is equal to the density topology $\mathcal{T}_d$, an opposite implication also holds. A proof is similar to that of Theorem 4 in [3].

Theorem 1. *For any $f \in \mathcal{A}$ we have*

(1) *if $\mathcal{T}_f \subset \mathcal{T}_d$ then $\limsup_{x\to 0+} \frac{f(x)}{x} < \infty$;*
(2) *if $\mathcal{T}_d \subset \mathcal{T}_f$ then $\liminf \frac{f(x)}{x} > 0$.*

Proof. Suppose that $\limsup_{x\to 0+} \frac{f(x)}{x} = \infty$. There is a sequence (a_n) such that $f(a_n) > na_n$ and $2a_{n+1} < a_n$ for $n \in \mathbb{N}$. Let

$$E := \bigcup_{n=1}^{\infty} [a_n, 2a_n].$$

For any $x \in (0, a_1)$ there is $n \in \mathbb{N}$ such that $x \in [a_n, a_{n-1})$. Hence

$$\frac{|E \cap (0,x)|}{f(x)} \leqslant \frac{|E \cap (0,2a_n)|}{f(a_n)} < \frac{2a_n}{f(a_n)}.$$

Since $\lim_{n\to\infty} \frac{2a_n}{f(a_n)} = 0$, $0 \in \Phi_f(\mathbb{R} \setminus E)$ and $\mathbb{R} \setminus E \in \mathcal{T}_f$, because $\mathbb{R} \setminus E$ is the union of an ordinary open set and $\{0\}$. However,

$$\limsup_{x\to 0^+} \frac{|E \cap (0,x)|}{x} \geqslant \limsup_{n\to\infty} \frac{|E \cap (0,2a_n)|}{2a_n} \geqslant \frac{1}{2},$$

and consequently $0 \notin \Phi_d(\mathbb{R} \setminus E)$. This gives $\mathbb{R} \setminus E \notin \mathcal{T}_d$, which completes the proof of (1).

Suppose now that $\liminf_{x\to 0+} \frac{f(x)}{x} = 0$. There is a sequence (a_n) such that $a_{n+1} < f(a_n) < \frac{1}{n} a_n$ for $n \in \mathbb{N}$. Let

$$E := \bigcup_{n=1}^{\infty} [a_n - f(a_n), a_n].$$

For each $x \in (0, a_1 - f(a_1))$ there exists $n \geqslant 2$ such that $x \in [a_n - f(a_n), a_{n-1} - f(a_{n-1}))$. Then

$$\frac{|E\cap(0,x)|}{x}<\frac{f(a_n)+a_{n+1}}{a_n-f(a_n)}<\frac{n}{n-1}\cdot\frac{f(a_n)+a_{n+1}}{a_n}<\frac{2}{n-1},$$

and so $\mathbb{R}\setminus E\in\mathcal{T}_d$. But

$$\limsup_{x\to 0^+}\frac{|E\cap(0,x)|}{f(x)}\geqslant\limsup_{n\to\infty}\frac{|E\cap(0,a_n)|}{f(a_n)}\geqslant 1,$$

and consequently $\mathbb{R}\setminus E\notin\mathcal{T}_f$. □

By Proposition 2 and Theorem 1, the family $\mathcal{A}$ falls naturally into four subfamilies.

Corollary 1. *Let $f\in\mathcal{A}$.*

(1) $\mathcal{T}_f=\mathcal{T}_d$ *if and only if* $0<\liminf_{x\to 0+}\frac{f(x)}{x}\leqslant\limsup_{x\to 0+}\frac{f(x)}{x}<\infty$.
(2) $\mathcal{T}_f\subsetneqq\mathcal{T}_d$ *if and only if* $0=\liminf_{x\to 0+}\frac{f(x)}{x}\leqslant\limsup_{x\to 0+}\frac{f(x)}{x}<\infty$.
(3) $\mathcal{T}_d\subsetneqq\mathcal{T}_f$ *if and only if* $0<\liminf_{x\to 0+}\frac{f(x)}{x}<\limsup_{x\to 0+}\frac{f(x)}{x}=\infty$.
(4) $\mathcal{T}_f\not\subseteq\mathcal{T}_d$ *and* $\mathcal{T}_d\not\subseteq\mathcal{T}_f$ *if and only if* $0=\liminf\frac{f(x)}{x}<\limsup_{x\to 0+}\frac{f(x)}{x}=\infty$.

The division described above is essential, when we examine properties of f-density topologies. Usually, one has used the division into two subfamilies: $\mathcal{A}^1:=\{f\in\mathcal{A}\colon \liminf_{x\to 0^+}\frac{f(x)}{x}>0\}$ and $\mathcal{A}^0:=\{f\in\mathcal{A}\colon \liminf_{x\to 0^+}\frac{f(x)}{x}=0\}$. In [2] it is proved that the inequality $\liminf_{x\to 0^+}\frac{f(x)}{x}>0$ implies that $\mathcal{T}_f$ is completely regular, and from $\liminf_{x\to 0^+}\frac{f(x)}{x}=0$ it follows that $\mathcal{T}_f$ is Hausdorff but not regular. In this paper we focus on the division into functions with $\limsup_{x\to 0+}\frac{f(x)}{x}<\infty$ (i.e. $\mathcal{T}_f\subset\mathcal{T}_d$) and with $\limsup_{x\to 0+}\frac{f(x)}{x}=\infty$ ($\mathcal{T}_f\not\subseteq\mathcal{T}_d$).

Comparising f-density topologies we will usually use the condition formulated in [4]. For any $f_1,f_2\in\mathcal{A}$ we define sequences (A_n), (ε_n) given by

$$A_n:=\left\{x\in(0,\infty)\colon\ f_1(x)<\frac{1}{n}f_2(x)\right\}\quad\text{and}\quad \varepsilon_n:=\limsup_{x\to 0+}\frac{|A_n\cap(0,x)|}{f_1(x)}.$$

Theorem 2. *([4, Theorem 4]) $\mathcal{T}_{f_1}\subset\mathcal{T}_{f_2}$ if and only if $\lim_{n\to\infty}\varepsilon_n=0$.*

In the sequel we usually will prove that, for given $f_1,f_2\in\mathcal{A}$, $\varepsilon_1=0$. Of course, in this case, other ε_n's are also equal to zero, and $\mathcal{T}_{f_1}\subset\mathcal{T}_{f_2}$.

2. Δ_2 condition for f-density topologies

The main role in our considerations will be played by the condition very similar to (Δ_2) condition considered in the theory of Orlicz spaces. Recall that a function $\varphi\colon[0,\infty)\to[0,\infty)$ is called a φ-function if it is continuous, nondecreasing and unbounded, with $\varphi(0)=0$, $\varphi(x)>0$ for $x>0$. W. Orlicz in [9] considered the family

$$\mathcal{M}:=\left\{f:[0,1]\to\overline{\mathbb{R}}\colon\ \int_0^1\varphi\big(\big|f(x)\big|\big)\,dx<\infty\right\}$$

and proved that $\mathcal{M}$ is a linear space if and only if φ satisfies the condition (Δ_2):

$$\limsup_{x\to\infty}\frac{\varphi(2x)}{\varphi(x)}<\infty.$$

Observe that (Δ_2) condition is useful in the theory of Orlicz spaces, Orlicz classes and Young functions. We remind some basic facts. Let φ be a φ-function and (Ω,Σ,μ) be a σ-finite complete measure space. Denote by X the space of all real-valued Σ-measurable and μ-a.e. finite functions on Ω, with equality a.e. If $x\in X$ then $\varphi(|x(t)|)$ is Σ-measurable and

$$\rho_\varphi(x):=\int_\Omega\varphi\big(\big|x(t)\big|\big)\,d\mu$$

is a left-continuous modular in X. The modular space

$$L_\varphi:=\left\{x\in X\colon\ \lim_{\lambda\to 0+}\rho_\varphi(\lambda x)=0\right\}$$

is called the Orlicz space. Let

$$L^0_\varphi := \{x \in X:\ \rho_\varphi(x) < \infty\} \quad \text{and} \quad L^a_\varphi := \{x \in X:\ \rho_\varphi(\lambda x) < \infty \text{ for every } \lambda > 0\}.$$

The set L^0_φ is called the Orlicz class.

Theorem 3. *(Cf. [8,11].) Suppose that μ is a nonatomic and finite measure. The following conditions are equivalent*:

(1) φ *satisfies* (Δ_2),
(2) $L^0_\varphi = L_\varphi$,
(3) $L^a_\varphi = L_\varphi$.

Note also that, assuming μ is separable, L_φ is separable if and only if φ satisfies (Δ_2) condition (see [8, Theorem 4.2] and [11, Theorem 3.5.1]).

Let f be a function from the family $\mathcal{A}$. Since, defining the topology generated by f, we examine the behaviour of a function for arguments close to zero, we can define an analogous condition:

Definition 1. We will say that a function f from $\mathcal{A}$ fulfills Δ_2 condition ($f \in \Delta_2$) if

$$\limsup_{x \to 0^+} \frac{f(2x)}{f(x)} < \infty.$$

It is useful to observe

Proposition 3. *For any $f \in \mathcal{A}$ the following conditions are equivalent*:

(1) $f \in \Delta_2$,
(2) *for any positive number* β, $\limsup_{x \to 0^+} \frac{f(\beta x)}{f(x)} < \infty$,
(3) *there exists* $\beta > 1$ *such that* $\limsup_{x \to 0^+} \frac{f(\beta x)}{f(x)} < \infty$.

Proof. (1) $\Rightarrow$ (2). If $\beta \leqslant 2$ then $f(\beta x) \leqslant f(2x)$ and the result is clear. If $\beta > 2$ then there is $n \in \mathbb{N}$ such that $2^n \geqslant \beta$, and from

$$\frac{f(\beta x)}{f(x)} \leqslant \frac{f(2^n x)}{f(x)} = \frac{f(2^n x)}{f(2^{n-1} x)} \cdot \frac{f(2^{n-1} x)}{f(2^{n-2} x)} \cdot \cdots \cdot \frac{f(2x)}{f(x)}$$

we obtain

$$\limsup_{x \to 0^+} \frac{f(\beta x)}{f(x)} \leqslant \left(\limsup_{x \to 0^+} \frac{f(2x)}{f(x)}\right)^n < \infty.$$

The implication (2) $\Rightarrow$ (3) is obvious and the proof of (3) $\Rightarrow$ (1) is analogous to the first one. $\square$

Observe that for any $\alpha \geqslant 1$, the function x^α (for $x > 0$) belongs to the family $\mathcal{A}$ and fulfills Δ_2 condition. Of course, $\mathcal{T}_{x^\alpha} \subset \mathcal{T}_d$. For $\alpha = 1$, $\mathcal{T}_x$ is equal to $\mathcal{T}_d$, for $\alpha = 2$ we obtain the superdensity topology (described in [7]) coarser than the density topology. The function

$$f(x) := \begin{cases} x^{\frac{1}{x}} & \text{for } x \in (0,1), \\ 1 & \text{for } x \in [1, \infty) \end{cases}$$

belongs to $\mathcal{A}$, it does not fulfill Δ_2, and the topology $\mathcal{T}_f$ is coarser than any $\mathcal{T}_{x^\alpha}$, $\alpha \geqslant 1$.

Note that functions x^α for $\alpha \in (0,1)$ do not belong to $\mathcal{A}$, because $\lim_{x \to 0^+} \frac{x^\alpha}{x} = \infty$. To obtain a function $f \in \Delta_2$, generating topology $\mathcal{T}_f$, which is finer than $\mathcal{T}_d$ (or uncomparable with $\mathcal{T}_d$), we will "glue together" square functions with various coefficients and constant ones. The method of such a construction is presented in the following lemma. It is worth to observe that the construction works not only for the square function. We can also use for example x^α with $\alpha > 1$.

Lemma 1. *If $(a_n)_{n \geqslant 0}$ is a decreasing sequence tending to zero and $b_n := \sqrt{a_n a_{n-1}}$ for $n \in \mathbb{N}$, then the functions*

$$f(x) := \begin{cases} \frac{x^2}{a_n} & \text{for } x \in [a_n, b_n], \\ a_{n-1} & \text{for } x \in [b_n, a_{n-1}], \\ a_0 & \text{for } x \geqslant a_0, \end{cases} \qquad g(x) := \begin{cases} \frac{x^2}{a_{2n-1}} & \text{for } x \in [b_{2n}, b_{2n-1}], \\ a_{2n} & \text{for } x \in [b_{2n+1}, b_{2n}], \\ a_0 & \text{for } x \geqslant b_1 \end{cases}$$

are continuous, they belong to $\mathcal{A}$ and fulfill Δ_2 condition.

Proof. Obviously, the functions f and g are continuous and belong to $\mathcal{A}$. Observe that

$$f(x) \leqslant \frac{x^2}{a_n} \quad \text{for } x \geqslant a_n.$$

Indeed, if $x \in [a_k, b_k]$ for some $k \leqslant n$, then $f(x) = \frac{x^2}{a_k} \leqslant \frac{x^2}{a_n}$, whereas for $x \in [b_k, a_{k-1}]$, $k \leqslant n$, we have $f(x) = f(b_k) = \frac{b_k^2}{a_k} \leqslant \frac{x^2}{a_n}$.

Let $x > 0$. If $x \in [a_n, b_n]$ for some n, then $\frac{f(2x)}{f(x)} \leqslant \frac{(2x)^2/a_n}{x^2/a_n} = 4$. If $x, 2x \in [b_n, a_{n-1})$ then $\frac{f(2x)}{f(x)} = 1$. Suppose now that $x \in [b_n, a_{n-1}]$ and $2x > a_{n-1}$. Then $b_n \leqslant x \leqslant a_{n-1} < 2x$, and using the previous inequalities we obtain

$$\frac{f(2x)}{f(x)} = \frac{f(2x)}{f(a_{n-1})} \cdot \frac{f(a_{n-1})}{f(x)} \leqslant \frac{f(2a_{n-1})}{f(a_{n-1})} \cdot 1 \leqslant 4.$$

Thus $\frac{f(2x)}{f(x)} \leqslant 4$ for all $x > 0$, and f fulfills Δ_2 condition.

Analogously,

$$g(x) \leqslant \frac{x^2}{a_{2n-1}} \quad \text{for } x \geqslant b_{2n}.$$

Indeed, if $x \in [b_{2k}, b_{2k-1}]$ for some $k \leqslant n$, then $g(x) = \frac{x^2}{a_{2k-1}} \leqslant \frac{x^2}{a_{2n-1}}$, whereas for $x \in [b_{2k-1}, b_{2k-2}]$, $k \leqslant n$, we have $g(x) = g(b_{2k-1}) = \frac{b_{2k-1}^2}{a_{2k-1}} \leqslant \frac{x^2}{a_{2n-1}}$.

Let $x > 0$. If $x \in [b_{2n}, b_{2n-1}]$ for some n, then $\frac{g(2x)}{g(x)} \leqslant \frac{(2x)^2/a_{2n-1}}{x^2/a_{2n-1}} = 4$. If $x, 2x \in [b_{2n+1}, b_{2n}]$ then $\frac{g(2x)}{g(x)} = 1$. Suppose now that $x \in [b_{2n+1}, b_{2n}]$ and $2x > b_{2n}$. Then $b_{2n+1} \leqslant x \leqslant b_{2n} < 2x$, and using the previous inequalities we obtain

$$\frac{g(2x)}{g(x)} = \frac{g(2x)}{g(b_{2n})} \cdot \frac{g(b_{2n})}{g(x)} \leqslant \frac{g(2b_{2n})}{g(b_{2n})} \cdot 1 \leqslant 4.$$

Consequently, $g \in \Delta_2$. $\square$

3. Algebraic properties of *f*-density topologies

Immediately from the definition of f-density it follows that if $E \in \mathcal{T}_f$ and $x \in \mathbb{R}$ then $E + x \in \mathcal{T}_f$ and $-E \in \mathcal{T}_f$, i.e. the topology $\mathcal{T}_f$ is invariant with respect to translations and point symmetries. It is easily seen that

Proposition 4. *(See [1].) If $\alpha \geqslant 1$ and $E \in \mathcal{T}_f$, then $\alpha E \in \mathcal{T}_f$.*

Thus all f-density topologies are invariant under multiplication by numbers $\alpha \geqslant 1$. In [1, Proposition 5] it has been shown that there exist f-density topologies which are not invariant under multiplication by $\alpha \in (0, 1)$. We will try to check, which f-density topologies are invariant under multiplication by every nonzero number.

Theorem 4. *If a function f from $\mathcal{A}$ fulfills Δ_2 condition, then $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers.*

Proof. According to Propositions 1 and 4, it is enough to show that for any $\alpha \in (0, 1)$ and $E \in \mathcal{L}$, from $0 \in \Phi_f^+(E)$ it follows that $0 \in \Phi_f^+(\alpha E)$. Suppose that $\alpha \in (0, 1)$ and $\lim_{x \to 0^+} \frac{|E' \cap (0,x)|}{f(x)} = 0$. Hence

$$\frac{|(\alpha E)' \cap (0, x)|}{f(x)} = \frac{\alpha |E' \cap (0, \frac{x}{\alpha})|}{f(\frac{x}{\alpha})} \cdot \frac{f(\frac{x}{\alpha})}{f(x)},$$

and by Proposition 3

$$\limsup_{x \to 0^+} \frac{|(\alpha E)' \cap (0, x)|}{f(x)} \leqslant \alpha \cdot \limsup_{x \to 0^+} \frac{|E' \cap (0, \frac{x}{\alpha})|}{f(\frac{x}{\alpha})} \cdot \limsup_{x \to 0^+} \frac{f(\frac{x}{\alpha})}{f(x)} = 0. \qquad \square$$

We will show that if $\mathcal{T}_f \subset \mathcal{T}_d$ then Theorem 4 can be reversed, i.e. the invariantness of $\mathcal{T}_f$ under multiplication by nonzero numbers implies $f \in \Delta_2$. In [12, Theorem 2.8] is proved a similar result for functions of the form $f(x) = x\psi(x)$, where ψ is a nondecreasing continuous function with $\lim_{x \to 0+} \psi(x) = 0$, but the construction is rather complicated. Fortunately, our general framework allows us to simplify a proof. The following lemma plays a fundamental role in our considerations.

Lemma 2. *Suppose that $\mathcal{T}_f \subset \mathcal{T}_d$ and $\limsup_{x\to 0^+} \frac{f(\alpha x)}{f(x)} = \infty$. There exists an interval set $E := \bigcup_{n=1}^{\infty}[a_n, b_n]$ such that zero is not an f-dispersion point of E but it is an f-dispersion point of the set $\alpha^2 E$.*

Proof. From Theorem 1 we know that there are positive numbers M and h such that $\frac{f(x)}{x} \leqslant M$ for $x \in (0, h)$. From the assumption it follows that $\alpha > 1$ and there is a decreasing sequence (b_n) such that $b_1 < h$, $b_{n+1} < \min\{\frac{b_n}{\alpha}, \frac{f(b_n)}{2}\}$ and $f(\alpha b_n) > 2^n f(b_n)$ for $n \in \mathbb{N}$. Let

$$a_n := \max\left\{\frac{b_n}{\alpha}, b_n - \frac{f(b_n)}{2}\right\}$$

and

$$E := \bigcup_{n=1}^{\infty}[a_n, b_n].$$

Then $b_n - a_n = \min\{\frac{f(b_n)}{2}, b_n - \frac{b_n}{\alpha}\}$ and

$$\frac{|E \cap (0, b_n)|}{f(b_n)} > \frac{b_n - a_n}{f(b_n)} \geqslant \min\left\{\frac{1}{2}, \frac{1}{M}\left(1 - \frac{1}{\alpha}\right)\right\}.$$

Consequently, zero is not an f-dispersion point of E.

Observe that $\alpha^2 a_n \geqslant \alpha b_n$ for $n \in \mathbb{N}$. For $x \in [\alpha^2 a_n, \alpha^2 b_n]$ we have

$$\frac{|\alpha^2 E \cap (0, x)|}{f(x)} < \frac{\alpha^2(b_{n+1} + (b_n - a_n))}{f(\alpha b_n)} < \frac{\alpha^2 f(b_n)}{f(\alpha b_n)} < \frac{\alpha^2}{2^n}.$$

Using the preceding inequality, for $x \in [\alpha^2 b_{n+1}, \alpha^2 a_n]$ we obtain

$$\frac{|\alpha^2 E \cap (0, x)|}{f(x)} = \frac{|\alpha^2 E \cap (0, \alpha^2 b_{n+1})|}{f(x)} \leqslant \frac{|\alpha^2 E \cap (0, \alpha^2 b_{n+1})|}{f(\alpha^2 b_{n+1})} < \frac{\alpha^2}{2^{n+1}}.$$

Therefore zero is an f-dispersion point of $\alpha^2 E$. □

Theorem 5. *Suppose that $\mathcal{T}_f \subset \mathcal{T}_d$. The topology $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers if and only if $f \in \Delta_2$.*

Proof. Theorem 4 and Proposition 4 imply that it suffices to show that for any $f \notin \Delta_2$ and $\gamma \in (0, 1)$ there is a set $A \in \mathcal{T}_f$ such that $\gamma A \notin \mathcal{T}_f$. Fix $f \notin \Delta_2$ and $\gamma \in (0, 1)$. From Proposition 3 it follows that $\limsup_{x\to 0+} \frac{f(x/\sqrt{\gamma})}{f(x)} = \infty$. Let E be the set defined in Lemma 2 for $\alpha := \frac{1}{\sqrt{\gamma}}$ and write $A := \mathbb{R} \setminus \frac{1}{\sqrt{\gamma}}E$. Then $A \in \mathcal{T}_f$ and $\gamma A \notin \mathcal{T}_f$, which finishes the proof. □

The above theorem can be reversed, i.e. if $\mathcal{T}_f \not\subseteq \mathcal{T}_d$ then the invariantness of $\mathcal{T}_f$ under multiplication by nonzero numbers is not equivalent to $f \in \Delta_2$.

Theorem 6. *If $\mathcal{T}_f \not\subseteq \mathcal{T}_d$ then there is a function $g \notin \Delta_2$ such that $\mathcal{T}_f = \mathcal{T}_g$.*

Proof. We may assume that $f \in \Delta_2$. Since $f \in \mathcal{A}$, $\liminf_{x\to 0+} \frac{f(x)}{x} < \infty$. Since $\mathcal{T}_f \not\subseteq \mathcal{T}_d$, using Proposition 2 we obtain $\limsup_{x\to 0+} \frac{f(x)}{x} = \infty$. Consequently, there are sequences (a_n), (b_n) and a positive number M such that for any $x \in (0, b_1]$ and $n \in \mathbb{N}$ we have $b_{n+1} < a_n < b_n$,

$$\frac{f(a_n)}{a_n} < M, \qquad \frac{f(b_n)}{b_n} > n^2 \quad \text{and} \quad \frac{f(2x)}{x} < M.$$

For any $n > M^2$ we set

$$c_n := \sup\{x \in [a_n, b_n]:\ f(x) \leqslant nb_n\}.$$

Thus

$$nb_n \geqslant f\left(\frac{c_n}{2}\right) \geqslant \frac{f(2c_n)}{M^2} > \frac{nb_n}{M^2},$$

$$f(2a_n) \leqslant Mf(a_n) < M^2 a_n < na_n < nb_n,$$

and $2a_n \leqslant c_n$. Let us define

$$g(x) := \begin{cases} f(b_n) & \text{for } x \in (\frac{c_n}{2}, b_n],\ n > M^2, \\ f(x) & \text{for } x \in \mathbb{R}^+ \setminus \bigcup_{n>M^2} (\frac{c_n}{2}, b_n]. \end{cases}$$

We have $g \notin \Delta_2$, because

$$\frac{g(c_n)}{g(\frac{c_n}{2})} = \frac{f(b_n)}{f(\frac{c_n}{2})} > \frac{n^2 b_n}{n b_n} = n.$$

Since $f \leqslant g$, $\mathcal{T}_f \subset \mathcal{T}_g$. To prove the inverse inclusion, we use Theorem 2. Let

$$A_1 := \{x > 0\colon\ f(x) < g(x)\} \quad \text{and} \quad \varepsilon_1 := \limsup_{x\to 0+} \frac{|A_1 \cap (0,x)|}{f(x)}.$$

Since $A_1 \subset \bigcup_{n>M^2} (\frac{c_n}{2}, b_n]$, for any $x \in (\frac{c_n}{2}, \frac{c_{n-1}}{2}]$ we have

$$\frac{|A_1 \cap (0,x)|}{f(x)} < \frac{b_n}{f(\frac{c_n}{2})} < \frac{M^2}{n},$$

and consequently $\varepsilon_1 = 0$. Thus $\mathcal{T}_g \subset \mathcal{T}_f$. $\square$

Corollary 2. *Let $f \in \mathcal{A}$. The following conditions are equivalent.*

(1) $\mathcal{T}_f \subset \mathcal{T}_d$.
(2) *The topology $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers if and only if $f \in \Delta_2$.*

We still should show that there exists an f-density topology which is invariant under multiplication by nonzero numbers and is not included in the density topology. It is sufficient to find a function f fulfilling Δ_2 condition such that $\mathcal{T}_f \not\subset \mathcal{T}_d$.

Example 1. Let $(a_n)_{n\geqslant 0}$ be a decreasing sequence tending to zero such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$ (for example $a_n := \frac{1}{(n+1)!}$) and let f, g be the functions defined in Lemma 1, i.e.

$$f(x) := \begin{cases} \frac{x^2}{a_n} & \text{for } x \in [a_n, b_n], \\ a_{n-1} & \text{for } x \in [b_n, a_{n-1}], \\ a_0 & \text{for } x \geqslant a_0, \end{cases} \qquad g(x) := \begin{cases} \frac{x^2}{a_{2n-1}} & \text{for } x \in [b_{2n}, b_{2n-1}], \\ a_{2n} & \text{for } x \in [b_{2n+1}, b_{2n}], \\ a_0 & \text{for } x \geqslant b_1 \end{cases}$$

where $b_n := \sqrt{a_n a_{n-1}}$ for $n \in \mathbb{N}$. Then $f, g \in \Delta_2$, and so the topologies $\mathcal{T}_f$ and $\mathcal{T}_g$ are invariant under multiplication by nonzero numbers. Since $\liminf_{x\to 0+} \frac{f(x)}{x} = 1$ and

$$\limsup_{x\to 0+} \frac{f(x)}{x} \geqslant \limsup_{n\to\infty} \frac{f(b_n)}{b_n} = \lim_{n\to\infty} \sqrt{\frac{a_{n-1}}{a_n}} = \infty,$$

Corollary 1 implies $\mathcal{T}_d \subsetneq \mathcal{T}_f$. Similarly, from $\limsup_{x\to 0+} \frac{g(x)}{x} = \infty$ and $\liminf_{x\to 0+} \frac{g(x)}{x} = 0$, we conclude that the topologies $\mathcal{T}_g$ and $\mathcal{T}_d$ are not comparable ($\mathcal{T}_g \not\subset \mathcal{T}_d$ and $\mathcal{T}_d \not\subset \mathcal{T}_g$).

Suppose that $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers. If $\mathcal{T}_f \subset \mathcal{T}_d$ then $\mathcal{T}_f$ can be generated only by functions fulfilling Δ_2 condition (Theorem 5). If $\mathcal{T}_f \not\subset \mathcal{T}_d$ then $\mathcal{T}_f$ is generated by some function which does not fulfill Δ_2 (Theorem 6). We do not know, if in this case, $\mathcal{T}_f$ has to be generated by some function from Δ_2.

Problem 1. Suppose that $\mathcal{T}_f \not\subset \mathcal{T}_d$ and $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers. Does there exist a function $g \in \Delta_2$ such that $\mathcal{T}_f = \mathcal{T}_g$?

It is known that every f-density topology is generated by some function g, which is constant on intervals, i.e. $g(x) := b_n$ for $x \in (a_{n+1}, a_n]$, where (a_n) and (b_n) are decreasing sequences tending to zero (cf. [3, Theorem 1]).

Let $\langle a\rangle := (a_n)$ be a decreasing sequence tending to zero and

$$f_{\langle a\rangle}(x) := \begin{cases} a_n & \text{for } x \in (a_{n+1}, a_n],\ n \in \mathbb{N}, \\ a_1 & \text{for } x > a_1. \end{cases}$$

Obviously, $f_{\langle a\rangle} \in \mathcal{A}$ and $f_{\langle a\rangle}(x) \geqslant x$ for every x, so $\mathcal{T}_d \subset \mathcal{T}_{f_{\langle a\rangle}}$. In [3, Theorem 5], it has been proved that density-type topologies generated by functions of the form $f_{\langle a\rangle}$ are the same as density-type topologies generated by sequences (cf. [6]

and [5]). The problem of existence of an f-density topology which is finer than the density topology $\mathcal{T}_d$, and is not a topology generated by a sequence, has been solved in [3, Theorem 6]. But constructed in this paper function $f_0 \in \mathcal{A}$ with $\mathcal{T}_d \subset \mathcal{T}_{f_0}$ and $\mathcal{T}_{f_0} \neq \mathcal{T}_{f_{\langle a\rangle}}$ for any decreasing and tending to zero sequence $\langle a\rangle$, is very complicated. Now we can show an easier solution.

Example 2. Let f be the function from Example 1. We have proved that $\mathcal{T}_d \subsetneqq \mathcal{T}_f$ and $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers. Theorems 3 and 4 from [6] show that, if $\mathcal{T}_d \subsetneqq \mathcal{T}_{f_{\langle a\rangle}}$ then $\mathcal{T}_{f_{\langle a\rangle}}$ cannot be invariant under multiplication by any $m \in (0,1)$. Thus for any decreasing and tending to zero sequence $\langle a\rangle$, $\mathcal{T}_f \neq \mathcal{T}_{f_{\langle a\rangle}}$.

Problem 2. Assume that $\mathcal{T}_d \subsetneqq \mathcal{T}_f$ and $\mathcal{T}_f$ is different from any $\mathcal{T}_{f_{\langle a\rangle}}$. Must $\mathcal{T}_f$ be invariant under multiplication by nonzero numbers?

4. Comparison of *f*-density topologies

A direct calculation shows that the equality $\lim_{x\to 0+} \frac{f_1(x)}{f_2(x)} = \infty$ leads to $\mathcal{T}_{f_2} \subsetneqq \mathcal{T}_{f_1}$ (cf. [4, Corollary 1]). We can easily obtain the same conclusion from Theorem 2. At the beginning of the paper we have mentioned that the condition $\limsup_{x\to 0+} \frac{f_1(x)}{f_2(x)} < \infty$ implies $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$. We have also constructed functions $\widetilde{f}_1$ and $\widetilde{f}_2$ such that $\limsup_{x\to 0+} \frac{\widetilde{f}_1(x)}{\widetilde{f}_2(x)} = \infty$ and $\mathcal{T}_{\widetilde{f}_1} = \mathcal{T}_{\widetilde{f}_2}$. It is easy to check that $\widetilde{f}_1, \widetilde{f}_2 \notin \Delta_2$.

Our next goal is to compare f-density topologies fulfilling Δ_2. We will show that there exist functions $f_1, f_2 \in \Delta_2$ such that $\limsup_{x\to 0+} \frac{f_2(x)}{f_1(x)} = \infty$ and $\mathcal{T}_{f_1} = \mathcal{T}_{f_2}$. However, if we assume that $\mathcal{T}_{f_1} \subset \mathcal{T}_d$, then $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$ implies $\limsup_{x\to 0+} \frac{f_1(x)}{f_2(x)} < \infty$.

Example 3. We will define functions $f_1, f_2 \in \Delta_2$ such that $\limsup_{x\to 0+} \frac{f_2(x)}{f_1(x)} = \infty$ and $\mathcal{T}_{f_1} = \mathcal{T}_{f_2}$. Fix $a \in (0,1)$ and set

$$f_1(x) := \begin{cases} x^2 a^{-5\cdot 5^n} & \text{for } x \in [a^{5\cdot 5^n}, a^{3\cdot 5^n}],\ n = 0,1,\ldots, \\ a^{5^n} & \text{for } x \in [a^{3\cdot 5^n}, a^{5^n}],\ n = 0,1,\ldots, \\ a & \text{for } x \geqslant 1, \end{cases}$$

$$f_2(x) := \begin{cases} x^2 a^{-5\cdot 5^n} & \text{for } x \in [a^{5\cdot 5^n}, a^{4\cdot 5^n}],\ n = 0,1,\ldots, \\ x^3 a^{-9\cdot 5^n} & \text{for } x \in [a^{4\cdot 5^n}, a^{\frac{10}{3}\cdot 5^n}],\ n = 0,1,\ldots, \\ a^{5^n} & \text{for } x \in [a^{\frac{10}{3}\cdot 5^n}, a^{5^n}],\ n = 0,1,\ldots, \\ a & \text{for } x \geqslant 1. \end{cases}$$

Obviously, f_1, f_2 are continuous, they belong to $\mathcal{A}$ and $f_1(x) = f_2(x)$ for $x \notin \bigcup_{n=0}^{\infty} (a^{4\cdot 5^n}, a^{\frac{10}{3}\cdot 5^n})$. If $x \in (a^{4\cdot 5^n}, a^{\frac{10}{3}\cdot 5^n})$ then $\frac{f_2(x)}{f_1(x)} = \frac{x}{a^{4\cdot 5^n}} > 1$, so $f_2(x) > f_1(x)$. Consequently, $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$. To prove the inverse inclusion we use Theorem 2. Let

$$A_1 := \{x > 0\colon\ f_1(x) < f_2(x)\} \quad \text{and} \quad \varepsilon_1 := \limsup_{x\to 0+} \frac{|A_1 \cap (0,x)|}{f_1(x)}.$$

Since $A_1 \subset \bigcup_{n=0}^{\infty} (a^{4\cdot 5^n}, a^{\frac{10}{3}\cdot 5^n})$, for any $x \in (a^{4\cdot 5^{n+1}}, a^{4\cdot 5^n})$ we have

$$\frac{|A_1 \cap (0,x)|}{f_1(x)} < \frac{a^{\frac{10}{3}\cdot 5^{n+1}}}{f_1(a^{4\cdot 5^{n+1}})} = a^{\frac{1}{3}\cdot 5^{n+1}}.$$

Hence $\varepsilon_1 = 0$, and so $\mathcal{T}_{f_2} \subset \mathcal{T}_{f_1}$. Lemma 1 implies $f_1 \in \Delta_2$. The proof that $f_2 \in \Delta_2$ is similar to the proof of Lemma 1. Finally observe that

$$\limsup_{x\to 0^+} \frac{f_2(x)}{f_1(x)} \geqslant \limsup_{n\to\infty} \frac{f_2(a^{\frac{10}{3}\cdot 5^n})}{f_1(a^{\frac{10}{3}\cdot 5^n})} = \lim_{n\to\infty} a^{-\frac{2}{3}\cdot 5^n} = \infty.$$

Theorem 7. *Suppose that $f \in \Delta_2$ and $\mathcal{T}_f \subset \mathcal{T}_d$. If $\mathcal{T}_f \subset \mathcal{T}_g$ then $\limsup_{x\to 0+} \frac{f(x)}{g(x)} < \infty$.*

Proof. Suppose, contrary to our claim, that $\limsup_{x\to 0^+} \frac{f(x)}{g(x)} = \infty$. Theorem 1(1) and the definition of Δ_2 condition imply that there are positive numbers L and ε such that for any $x \in (0,\varepsilon)$,

$$\frac{f(x)}{x} < L \quad \text{and} \quad \frac{f(2x)}{f(x)} < L.$$

Since $\limsup_{x\to 0+}\frac{f(x)}{g(x)}=\infty$, there exists a decreasing sequence (b_n) with $b_1<\varepsilon$, such that

$$\frac{f(b_n)}{g(b_n)}>(L+n)^2 \quad\text{and}\quad b_{n+1}<\min\left\{\frac{b_n}{2},\frac{g(b_n)}{2}\right\}.$$

Observe that

$$\frac{b_n}{g(b_n)}=\frac{b_n}{f(b_n)}\cdot\frac{f(b_n)}{g(b_n)}>\frac{(L+n)^2}{L}>L,$$

and that for $x\in[\frac{b_n}{2},b_n]$,

$$\frac{f(x)}{g(x)}\geqslant\frac{f(\frac{b_n}{2})}{g(b_n)}>\frac{f(b_n)}{L\cdot g(b_n)}>L+n.$$

Set $a_n:=\max\{\frac{b_n}{2},b_n-\frac{g(b_n)}{2}\}$ and

$$E:=\bigcup_{n=1}^{\infty}[a_n,b_n].$$

Since

$$\frac{|E\cap(a_n,b_n)|}{g(b_n)}\geqslant\frac{b_n-a_n}{g(b_n)}=\frac{\min\{\frac{b_n}{2},\frac{g(b_n)}{2}\}}{g(b_n)}\geqslant\min\left\{\frac{L}{2},\frac{1}{2}\right\},$$

zero is not a g-dispersion point of E, and consequently $\mathbb{R}\setminus E\notin\mathcal{T}_g$.

On the other hand, for $x\in[a_n,b_n]$ we have

$$\frac{|E\cap(0,x)|}{f(x)}\leqslant\frac{b_{n+1}+(b_n-a_n)}{f(\frac{b_n}{2})}<\frac{g(b_n)}{f(b_n)}\cdot\frac{f(b_n)}{f(\frac{b_n}{2})}<\frac{L}{(L+n)^2}.$$

If $x\in(b_{n+1},a_n)$, then $|E\cap(0,x)|=|E\cap(0,b_{n+1})|$ and, using the latter result, we obtain

$$\frac{|E\cap(0,x)|}{f(x)}\leqslant\frac{|E\cap(0,b_{n+1})|}{f(b_{n+1})}<\frac{L}{(L+n+1)^2}.$$

Therefore $0\in\Phi_f(\mathbb{R}\setminus E)$ and $\mathbb{R}\setminus E\in\mathcal{T}_f$, contrary to $\mathcal{T}_f\subset\mathcal{T}_g$. $\square$

References

[1] M. Filipczak, T. Filipczak, A generalization of the density topology, Tatra Mt. Math. Publ. 34 (2006) 37–47.
[2] M. Filipczak, T. Filipczak, On f-density topologies, Topology Appl. 155 (2008) 1980–1989.
[3] M. Filipczak, T. Filipczak, Density topologies generated by functions and by sequences, Tatra Mt. Math. Publ. 40 (2008) 103–115.
[4] M. Filipczak, T. Filipczak, On the comparison of density type topologies generated by functions, Real Anal. Exchange 36 (2010–2011) 341–352.
[5] M. Filipczak, T. Filipczak, J. Hejduk, On the comparison of the density type topologies, Atti Semin. Mat. Fis. Univ. Modena 52 (2004) 37–46.
[6] M. Filipczak, J. Hejduk, On topologies associated with the Lebesgue measure, Tatra Mt. Math. Publ. 28 (2004) 187–197.
[7] J. Lukeš, J. Malý, L. Zajíček, Fine Topology Methods in Real Analysis and Potential Theory, Springer-Verlag, 1986.
[8] L. Maligranda, Orlicz Spaces and Interpolation, Sem. Mat., Univ. Estadual de Campinas, 1989.
[9] W. Orlicz, Über eine gewisse Klasse von Räumen vom typus B, Bull. Acad. Pol., Sér. A (1932) 207–220; reprinted in his Collected Papers, PWN, Warszawa, 1988, pp. 217–230.
[10] J.C. Oxtoby, Measure and Category, Springer-Verlag, 1980.
[11] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Marcel Dekker, Inc., 1991.
[12] M. Terepeta, E. Wagner-Bojakowska, ψ-Density topology, Rend. Circ. Mat. Palermo (2) 48 (1999) 451–476.
[13] E. Wagner-Bojakowska, W. Wilczyński, Comparison of ψ-density topologies, Real Anal. Exchange 25 (1999–2000) 661–672.
[14] W. Wilczyński, Density topologies, in: Handbook of Measure Theory, Elsevier Science B.V./North-Holland, Amsterdam, 2002, pp. 675–702.